Some Notes on the Optimum Design of Stepped Transmission-Line Transformers*

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Summary—**This paper describes an optimum design of monotonic stepped transmission-line transformers when the reflection coefficient and the bandwidth ratio are prescribed. For the analysis, discontinuity capacitances and reflection interactions are neglected and the validity of the conclusions is therefore restricted to small steps. The analysis is applicable to a multistep transmission line of which the quarter-wave transformer is a special case. In particular, it is shown that if the number of steps is increased from three to five a larger bandwidth may be obtained, but it is not possible to reduce the over-all length in this manner. For a given bandwidth, the shortest taper is always a stepped transmission line and never a continuous one.**

Introduction

THE optimum design of stepped transmission-line transformers (hence called step-lines) has been a subject of interest for the past few years.

At first Burkhardtmaier[1] gave an optimum design method for step-lines using the Tchebycheff polynomials. Bolinder,[2] solving approximately the synthesis of continuous lines, suggested the use of Dolph's method for optimizing the properties of the step-lines. Collin,[3] independently of Burkhardtmaier, solved the same problem and got the same results. Riblet[4] gave the general synthesis of step-lines and proved the physical realizability of the optimum step-line. Cohn,[5] supposing small steps, derived simple expressions for the design of the optimum step-line.

The problem solved in these papers is the following: For a specified number of steps the design method provides the maximum possible bandwidth for a specified reflection coefficient, or conversely, the minimum possible reflection coefficient for a given bandwidth. The length of the steps (at center frequency) is in each case a quarter-wavelength.

By increasing the number of steps, wider bandwidth is attainable, but the over-all length of the step-line increases. The question obviously arises: Is it possible to realize the same specified reflection coefficient with a shorter over-all length (using, where necessary, steps shorter than the quarter-wavelength) if we allow a certain reduction in the bandwidth?

Because of the validity of the different approximations, some restrictions upon the form of the step-line are necessary. (For the design, all the solutions neglect the higher order modes and the junction discontinuities.) Therefore we use the requirement that the taper has to be monotonic. In our calculations we shall apply the same approximations as Cohn; *i.e.*, we assume small steps.

Design of a General (Not-Quarter-Wave) Step-Line

Assuming that the steps are small, we may neglect reflection interactions and express the reflection coefficient of the step-line (see Fig. 1) referred to the center as follows:

$$\rho = A_1 e^{j(n-1)\phi} + A_2 e^{j(n-3)\phi} + \cdots + A_n e^{-j(n-1)\phi} \quad (1)$$

where

$$A_m = \frac{Z_{m+1} - Z_m}{Z_{m+1} + Z_m} \simeq \frac{1}{2} \ln \frac{Z_{m+1}}{Z_m},$$

Z_m = the characteristic impedance of the mth step,
$\phi = \beta l$,
β = phase-change coefficient,
l = the length of a step.

Supposing the step reflections to be symmetrical, *i.e.*, $A_1 = A_n$, $A_2 = A_{n-1}$, etc., and n to be odd (as will be clear later, this by no means restricts generality), we get

$$\rho = A_{(n+1)/2} + 2A_{(n-1)/2} \cos 2\phi + 2A_{(n-3)/2} \cos 4\phi + \cdots + 2A_1 \cos (n-1)\phi. \quad (2)$$

To get an optimum performance, we let the reflection coefficient be proportional to a Tchebycheff polynomial, *i.e.*,

$$\rho = \alpha T_{(n-1)/2}(\mu \cos 2\phi + \nu) \quad (3)$$

where[6]

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[1] W. Burkhardtmaier, "Widerstandstransformation mit Leitungen," *Funk und Ton*, vol. 3, pp. 151–167, 202–213; March, 1949.

[2] F. Bolinder, "Fourier transforms in the theory of inhomogeneous transmission lines," in "Kungliga Tekniska Hogskolans Handlingar," Stockholm, Sweden, no. 48; 1951.

[3] R. E. Collin, "Theory and design of wide-band multi-section quarter wave transformers," Proc. IRE, vol. 43, pp. 179–185; February, 1955.

[4] H. J. Riblet, "General synthesis of quarter-wave impedance transformers," IRE Trans. on Microwave Theory and Techniques, vol. MTT-5, pp. 36–37; January, 1957.

[5] S. B. Cohn, "Optimum design of stepped transmission-line transformers," IRE Trans. on Microwave Theory and Techniques, vol. MTT-3, pp. 16–21; April, 1955.

[6] We use the same method by which Riblet generalized Dolph's paper. See H. J. Riblet, "Discussion on 'A current distribution for broadside arrays which optimizes the relationships between beam-width and side-lobe level,' " Proc. IRE, vol. 35, pp. 489–492; May, 1947.

C. L. Dolph, "A current distribution for broadside arrays which optimizes the relationship between beam-width and side-lobe level," Proc. IRE, vol. 34, pp. 335–348; June, 1946.

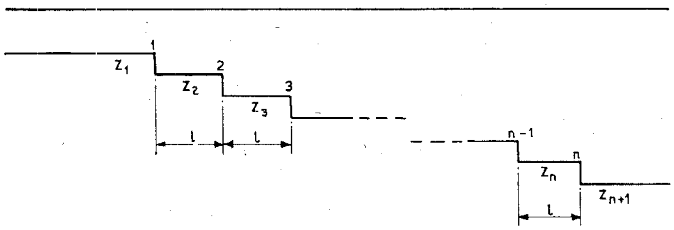

Fig. 1—The stepped transmission line.

$$T_m(x) = \cos (m \cos^{-1} x) \qquad |x| \le 1$$
$$T_m(x) = \cosh (m \cosh^{-1} x) \qquad |x| \ge 1. \quad (4)$$

Now, having two free parameters μ and ν, we may prescribe the value of the reflection coefficient at zero frequency and at the highest frequency of the pass band as well. Hence the two unknowns in the argument may be determined by the conditions

$$\frac{\rho}{\rho_m} = p \quad \text{if} \quad \phi = 0 \quad (5)$$

and

$$\left|\frac{\rho}{\rho_m}\right| = 1 \quad \text{if} \quad \phi = \phi_2,$$

where ρ_m is the specified value of the reflection coefficient in the pass band and p is defined by (5); *i.e.*, it is the ratio of the reflection coefficient at zero frequency and the maximum reflection coefficient in the pass band. Choosing $\alpha = \rho_m$, the equations for the unknowns are

$$\mu + \nu = z_0$$
$$\mu \cos 2\phi_2 + \nu = -1 \quad (6)$$

where z_0 is to be determined by

$$T_{(n-1)/2}(z_0) = p. \quad (7)$$

Solving (6) for μ and ν, we get

$$\mu = \frac{z_0 + 1}{1 - \cos 2\phi_2}, \qquad \nu = -\frac{1 + z_0 \cos 2\phi_2}{1 - \cos 2\phi_2}. \quad (8)$$

In Fig. 2 we see the transformation

$$z = \mu \cos 2\phi + \nu. \quad (9)$$

When ϕ changes from 0 to $\phi_2(\phi_2 \le 90°)$, z runs from z_0 to -1. At $\phi = \phi_1$ (where ϕ_1 fulfills $\mu \cos 2\phi_1 + \nu = 1$) $z = 1$. So between ϕ_1 and ϕ_2, $|z| \le 1$.

Because of the properties of the Tchebycheff polynomials, if $|z| \le 1$, then $|T_{1/2(n-1)}(z)| \le 1$. So between $\lambda_2 = 2\pi l/\phi_2$ and $\lambda_1 = 2\pi l/\phi_1$ the reflection coefficient fulfills the requirements. The design is optimum in the sense that for a given p, and for a given ϕ_2 (*i.e.*, lower edge of the band is given), it results in the greatest bandwidth.

It may be shown that our design contains, as a special case, the design of Cohn. If $\phi_2 = 90$ degrees our design agrees with that of Cohn for n steps. If

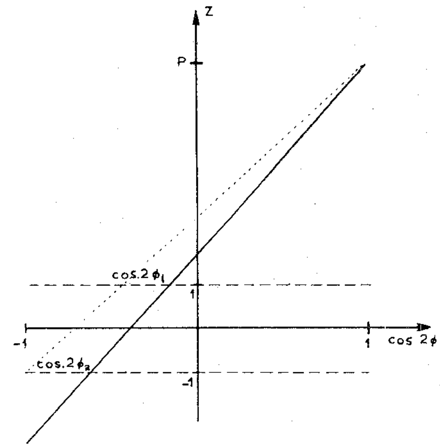

Fig. 2—The transformation $z = \mu \cos 2\phi + \nu$.

$$\phi_2 = \tfrac{1}{2} \cos^{-1}\left(-\frac{1}{z_0}\right),$$

i.e., $\nu = 0$, we get the step-line of Cohn for $(n+1)/2$ steps. So it also includes all the cases when the number of steps is even.

Further, we must compute the coefficients

$$A_k \left(\text{or the coefficient} \quad a_k = \frac{A_k}{\rho_m}.\right)$$

and with their help, determine the characteristic impedances. Performing the necessary calculations, we get for the ratio of consecutive characteristic impedances

$$\frac{Z_{m+1}}{Z_m} = \left(\frac{Z_{n+1}}{Z_1}\right)^{a_m/(a_1 + a_2 \cdots + a_n)}. \quad (10)$$

It may be immediately seen from (10) that the requirement for a monotonic step-line is equivalent to the mathematical condition that $a_m \ge 0$.

Examples

If $n = 3$, $T_{(n-1)/2}(z) = T_1(z) = z = \mu \cos 2\phi + \nu$.

Equating the coefficients

$$a_2 + 2a_1 \cos 2\phi = \mu \cos 2\phi + \nu \quad (11)$$

we get

$$a_2 = \nu; \quad a_1 = \tfrac{1}{2}\mu. \quad (12)$$

For the particular case of $p = 10$, Fig. 3(a) shows the value $L/\lambda_c[(L = (n-1))$ is the over-all length, λ_c is the wavelength at center frequency], and Fig. 3(d) the bandwidth ratio $(q = \lambda_1/\lambda_2)$ as a function of ϕ_2. It is seen that although the bandwidth is nearly constant, the over-all length can be made arbitrarily small. However the shortening of the taper will have a considerable influence on the characteristic impedances. Z_2 and Z_3 depend on μ and ν, which are plotted on Fig. 3(b) against ϕ_2. As ϕ_2 decreases, ν decreases also, becoming zero at $\phi_2 \cong 47.8$ degrees. This means that one step vanishes. The over-all length is just a quarter wavelength; *i.e.*, we have the ordinary quarter-wave transformer. If $\phi_2 < 47.8$ degrees, the over-all length will

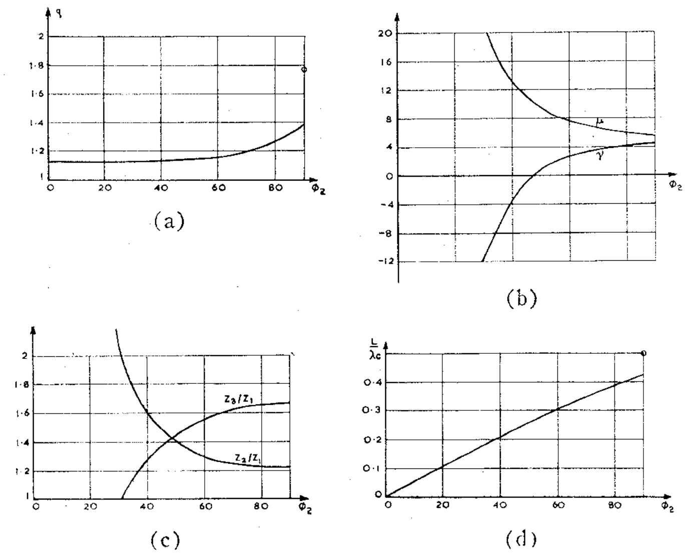

Fig. 3.

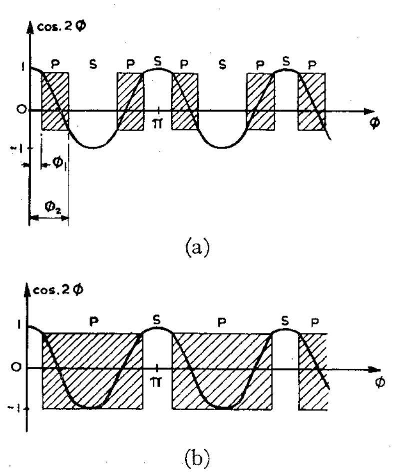

Fig. 4—The pass (p) and stop (s) bands, for (a) $\phi_2 < 90$ degrees, (b) $\phi_2 = 90$ degrees.

be shorter than a quarter-wavelength, but ν at the same time will become negative and the step-line is no longer monotonic. Supposing $Z_4/Z_1 = 2$, Fig. 3(c) shows Z_2/Z_1 and Z_3/Z_1 as functions of ϕ_2. At $\phi_2 \cong 47.8$ degrees, $Z_2 = Z_3 = \sqrt{2} Z_1$. If ϕ_2 decreases further, Z_3/Z_1 rapidly decreases and Z_2/Z_1 rapidly increases.

We must remark that the curves L/λ_c and q are discontinuous at $\phi_2 = 90$ degrees. The reason for this is shown in Fig. 4(a) and 4(b). Since the requirement for the pass band is $\cos 2\phi_2 \leq \cos 2\phi \leq \cos 2\phi_1$, we have an infinite number of pass and stop bands as shown on Fig. 4(a). If $\phi_2 = 90$ degrees, $\cos 2\phi_2 = -1$, and the first, second, third, and fourth, etc., pass bands merge into each other [Fig. 4(b)]. Consequently, the bandwidth, and the over-all length at the central frequency, suddenly increase.

If $n = 5$, $T_2(z) = 2z^2 - 1 = 2(\mu \cos 2\phi + \nu)^2 - 1$.

Equating the coefficients,

$$a_3 + 2a_2 \cos 2\phi + 2a_1 \cos 4\phi = \mu^2 \cos 4\phi + 4\mu \cos 2\phi + \mu^2 + 2\nu^2 - 1 \quad (13)$$

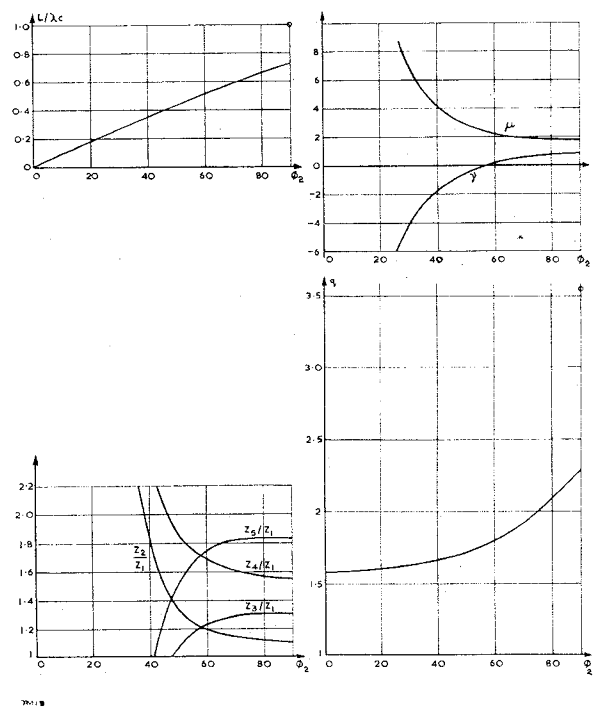

Fig. 5.

we get

$$a_3 = \mu^2 + 2\nu^2 - 1; \quad a_2 = 2\mu; \quad a_1 = \tfrac{1}{2}\mu^2. \quad (14)$$

The situation is similar again. If $\nu < 0$, the step-line is not monotonic. Fig. 5 shows the same quantities for $n = 5$, which we have drawn previously for $n = 3$. The consequences are also the same. If $\phi_2 \simeq 57.5$ degrees we get the same step-line as at $\phi_2 = 90$ degrees for $n = 3$.

Let us see now what is the connection between L/λ_c and q. Fig. 6 shows this curve for $p = 10$ and $n = 3, 5, 7, 9$. For any n the shortening of the step-line means a reduction of the bandwidth. At the points where L is an integral multiple of $\lambda_c/4$ (marked with small circles in Fig. 6), the number of steps in the step-line changes from n to $(n+1)/2$. The further shortening will result in a nonmonotonic step-line represented by the broken lines.

For the comparison of the actual lengths of the step-line for different values of n, the bandwidth ratio is plotted in Fig. 7 against L/λ_1. It may be seen that for a given p and q the length of this type step-line might be shorter than that of a quarter-wave type. If, for example, $p = 10$, $q = 1.85$, we get $n = 5$ and $L/\lambda_1 = 0.391$, while the usual (quarter-wave) design would result in $n = 4$ and $L/\lambda_1 = 0.412$.

The Shortening of the Step-Line by the Application of More Steps

We have seen that increasing the number of steps results in a larger bandwidth. Let us investigate the relation between the number of steps and over-all length for a given bandwidth and see if it is possible to

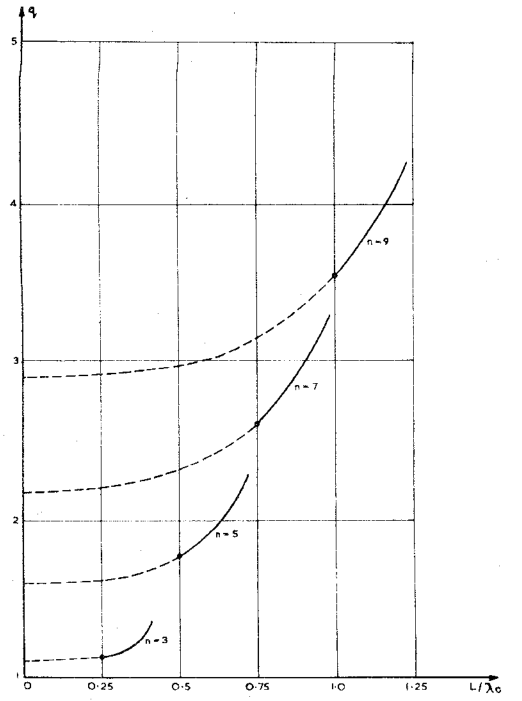

Fig. 6—The bandwidth ratio as a function of L/λ_1. The broken lines represent the nonmonotonic solutions.

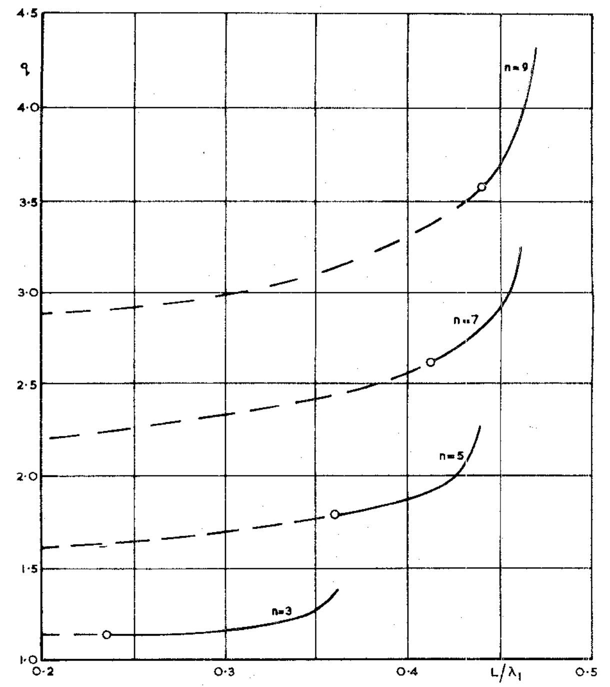

Fig. 7—The bandwidth ratio as a function of L/λ_1. The broken lines represent the nonmonotonic solutions.

shorten the monotonic step-line by using more steps. If this is possible, the problem can now be redefined: For a given reflection coefficient and bandwidth ratio, what is the optimum number of steps that will result in a minimum over-all length? Generally, to solve this problem is very difficult because of the complicated mathematical treatment involved. However a special case can be solved easily.

The normalized reflection coefficient for $n=5$ has the form

$$\frac{\rho}{\rho_m} = a_3 + 2a_2 \cos 2\phi + 2a_1 \cos 4\phi. \tag{15}$$

Let us introduce the notation $x = \cos 2\phi = \cos \beta L/2$, and arrange (15) in powers of x. Then

$$\frac{\rho}{\rho_m} = C_2 x^2 + C_1 x + C_0 \tag{16}$$

where

$$a_1 = \frac{C_2}{4}; \quad a_2 = \tfrac{1}{2}C_1; \quad a_3 = C_0 + \tfrac{1}{2}C_2. \tag{17}$$

If (16) has no linear term, *i.e.*, $a_2=0$, we get the special case of the step-line with three steps. When the linear term is not zero we get the step-line with five steps.

Let us first design a monotonic step-line for a given p and ϕ_2 whith three steps; and later, by adding the linear term, let us construct a shorter step-line with the same bandwidth but having five steps.

Prescribing $x_2 = \cos 2\phi_2$ the curve of the normalized reflection coefficient is shown in Fig. 8. It cuts the $\rho/\rho_m = 1$ line at the point $x = x_1$. These two points determine the bandwidth ratio and the length of the step-line. The nearer x_1 is to unity, the shorter is the step-line.

Thus, if we want to construct a shorter step-line having the same bandwidth, we have to construct a parabola through the points p, $x_1+\epsilon$, $x_2+\delta$ where ϵ and δ must satisfy

$$q = \frac{\cos^{-1} x_2}{\cos^{-1} x_1} = \frac{\cos^{-1}(x_2+\delta)}{\cos^{-1}(x_1+\epsilon)}. \tag{18}$$

It may be shown (see the Appendix), that C_1 is always negative; *i.e.*, these requirements cannot be fulfilled with a monotonic step-line of five steps. It is proved, therefore, that the application of two further steps cannot result in a shorter step-line.

Because of the mathematical difficulty of the general proof, we cannot generalize this result. From the physical point of view, however, it is very likely that if the application of two further steps does not result in a shorter step-line, then the application of any number of steps will not shorten it. Similarly, if the length of a three steps step-line cannot be shortened, then very likely the length of any properly designed step-line consisting of any number of steps cannot be reduced. Hence, if the bandwidth ratio and the reflection coefficient are specified, by drawing the diagrams similar to Fig. 7 we get the optimum number of steps.

Since in practice a finite bandwidth is always required, a finite number of steps always gives the shortest line. This statement does not mean that a continuous

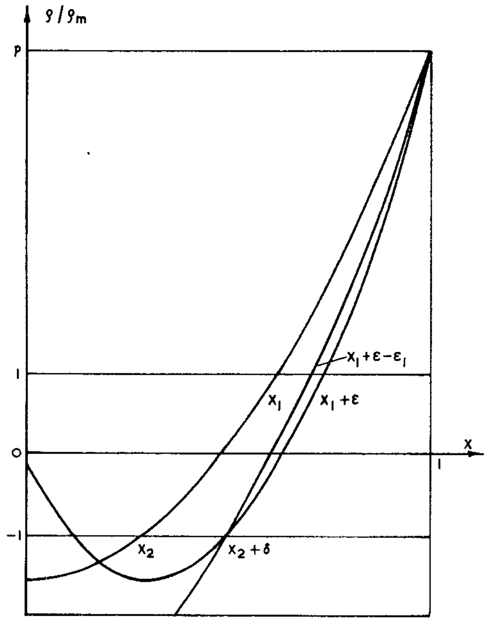

Fig. 8—The normalized reflection coefficient as a function of $x=\cos 2\phi$ for $n=3$ and $n=5$.

line (a step-line consisting of an infinite number of steps) cannot be useful. The application of a continuous taper may be recommended for microwaves if the resulting steps, and the discontinuity capacitances, are large, because the compensation of them will generally result in a narrower bandwidth.

There is always another application where the continuous line is preferable. This is the case when two or more pass bands are required. Because of the periodic structure of the reflection coefficient of step-lines, these requirements may be fulfilled with them only in special cases.

Appendix

The parabola for five steps through the points p, $x_1+\epsilon$, $x_2+\delta$ is shown in Fig. 8. The coefficient of the linear term is given by

$$C_1 = \frac{2-(p+1)(x_1+\epsilon)^2+(p-1)(x_2+\delta)^2}{[(x_1+\epsilon)-(x_2+\delta)][1-(x_1+\epsilon)][1-(x_2+\delta)]}. \quad (19)$$

Let us draw now a curve for three steps through the points $x_2+\delta$ and p. This curve cuts the $\rho/\rho_m=1$ line at the point $x_1+\epsilon-\epsilon_1$, where $\epsilon_1>0$[7] and the equation connecting $(x_2+\delta)$ and $x_1+\epsilon-\epsilon_1$ is as follows:

$$2-(p+1)(x_1+\epsilon)^2+(p-1)(x_2+\delta)^2 = -(p+1)\epsilon_1(2x_1+\epsilon-\epsilon_1). \quad (20)$$

We now prove that under the above conditions C_1 is negative. Since the denominator is positive it is sufficient to investigate the sign of the numerator.

Substituting (20) in the numerator of (19) we obtain

$$-(p+1)\epsilon_1[2(x_1+\epsilon)-\epsilon_1] \quad (21)$$

which is negative as far as

$$2(x_1+\epsilon)>\epsilon_1>0. \quad (22)$$

Hence C_1 is negative.

Acknowledgment

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[7] ϵ_1 must be positive, because the bandwidth ratio is decreasing as ϕ_2 decreases.

Microwave Semiconductor Switching Techniques*

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Summary—**This paper describes new microwave techniques employing the properties of N-type germanium diode switches. For applications requiring very high isolations, multiple switches are added in tandem. With proper spacing, they form antiresonant cavity circuits. In this case the isolations and insertion losses in db are directly additive. A switch is described which is normally ON and is pulsed OFF. Finally, details are given of a switch in a hybrid-tee configuration in which switching isolations of 50 db are obtained with an insertion loss of 0.7 db.**

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Introduction

In a previous publication,[1] a description is given of the low-power microwave semiconductor switch using N-type germanium. The switch consists of a germanium, point contact, diode placed across a section of standard X-band waveguide. Isolations of 25 to 35 db, with insertion losses of 1 db, are obtained over a 1000-mc bandwidth. The switching characteristics are

[1] M. A. Armistead, E. G. Spencer, and R. D. Hatcher, "Microwave semiconductor switch," Proc. IRE, vol. 44, p. 1875; December, 1956.